

Fluctuations and Noise: A General Model with Applications

R. F. O'Connell*

*Department of Physics and Astronomy, Louisiana State University,
Baton Rouge, Louisiana 70803-4001, USA*

(Dated: February 1, 2008)

Abstract

A wide variety of dissipative and fluctuation problems involving a quantum system in a heat bath can be described by the independent-oscillator (IO) model Hamiltonian. Using Heisenberg equations of motion, this leads to a generalized quantum Langevin equation (QLE) for the quantum system involving two quantities which encapsulate the properties of the heat bath. Applications include: atomic energy shifts in a blackbody radiation heat bath; solution of the problem of runaway solutions in QED; electrical circuits (resistively shunted Josephson barrier, microscopic tunnel junction, etc.); conductivity calculations (since the QLE gives a natural separation between dissipative and fluctuation forces); dissipative quantum tunneling; noise effects in gravitational wave detectors; anomalous diffusion; strongly driven quantum systems; decoherence phenomena; analysis of Unruh radiation and entropy for a dissipative system.

Keywords: Noise, fluctuations, dissipation, decoherence, Langevin equation, blackbody radiation, Unruh radiation

* E-mail: oconnell@phys.lsu.edu

Phone: (225) 578-6848 Fax: (225) 578-5855

I. INTRODUCTION

Noise is due to rapid fluctuations in the average of physical quantities. There are many types of classical noise (thermal, ...) but, in addition, there exists intrinsic quantum noise. A wide variety of noise problems can be described in a universal manner by means of a generalized quantum Langevin equation (QLE) [1]. This equation goes beyond the classical Langevin equation [2] by incorporating a potential, both quantum and non-Markovian effects, the quantum fluctuation-dissipation theorem, [3] as well as the presence of a time-dependent external force (thus allowing a description of the evolution of an irreversible system). Our purpose here is to outline how one obtains the QLE by starting with a very general microscopic model. Next, we consider a variety of applications. Before proceeding, we will first give a brief history of different approaches to fluctuation and dissipative processes.

As distinct from a dynamical system, a heat bath (reservoir) describes a subsystem with an infinite number of degrees of freedom which enable the system (i.e., the particle interacting with the heat bath) to relax in the course of time to a unique equilibrium state. A commonly used model of a heat bath is an infinite number of oscillators and the freedom of choice of the relevant masses and frequencies leads to a surprising diversity in the physical systems described, examples being blackbody radiation (BBR) and phonon heat baths and impurities in a metal (in the limit of the oscillator frequencies going to zero).

The beginning of the subject is generally regarded as having started with the observations of Robert Brown, a Scottish botanist who observed the random motion of pollen grains immersed in a fluid [4]. No external forces were present and the temperature T was room temperature. Later work showed that the very irregular motion occurred in any suspension of tiny particles in a liquid medium. The explanation of these phenomena was given in a series of papers by Einstein [5] from 1905 to 1908. Coupled with the experimental work of Perrin [6] this served to establish the atomic theory of matter since the irregular motion was clearly identified as being due to collisions with the molecules in the liquid. The term "Brownian motion" is now used in a generic sense to denote random motion and it covers a wide spectrum of phenomena from the motion of very fine particles suspended in gas to the motion of electrons in a BBR heat bath. *Einstein's* explanation of Brownian motion used a *discrete time* approach. In particular, his result that the diffusion constant D (which is

defined as one-half of the rate of change with time t of the mean-square displacement in the limit of large t is proportional to T/γ , where T is the temperature and γ^{-1} is the collision rate, is the first example of a fluctuation - dissipation (FD) theorem.

Shortly after the work of Einstein, Langevin [2] presented an entirely new approach to the subject which, in the words of Chandrasekhar [7], constitutes the "modern" approach to this and other such problems. The essence of *Langevin's* approach is a *continuous time* approach implemented by adoption of a *stochastic differential equation*, i.e., an equation for quantities which are *random* in nature. In other words, Langevin provided an elegant solution to the problem of generalizing a dynamical equation to a probabilistic equation. This was to be the start of a major new field of study with widespread implications in physics, chemistry, biology, and many other fields. The approach of Langevin was phenomenological in nature but its essential correctness has been verified by various microscopic studies. An essential feature of his approach was to separate the total force acting on a particle due to its environment into two parts: a frictional force and a fluctuation (random) force. These terms are very different in nature: The fluctuation term is basically microscopic in nature and has a time scale determined by the mean time between collisions whereas the time scale of the frictional force is proportional to the self-diffusion constant and is much larger.

Another example of a fluctuation-dissipation theorem was provided by Nyquist [8] who showed that the random fluctuations in voltage and current across a resistor measured by Johnson [9] are determined by its impedance (the famous Johnson-Nyquist noise in electrical circuits). All of the aforementioned work was classical in nature, but in 1951, Callen and Welton [3] presented their celebrated work on a quantum formulation of the FD theorem. Since a major shortcoming of the Langevin equation is its phenomenological nature, it is clearly desirable to obtain such an equation from microscopic considerations. Such a theory was given by Zwanzig [10] who considered the particle of interest to interact with an environment consisting of an infinite number of oscillators and obtained the usual classical high-temperature Brownian motion results. Turning to microscopic quantum dissipative phenomena, pioneering approaches to developing a quantum Langevin equation for particular problems appear in the articles of Senitzky [11] and Lax [12]. However, these articles were based on a Markovian approximation and, unfortunately, the work of Senitzky was marred by serious errors, as pointed out by Li et al.[13] Perhaps the most influential of the earlier articles is perhaps the oft-quoted article of Ford et al. [14] since it was the first

article in which the correct formulation of the quantum Langevin equation was indicated. Other important contributions include the work of Mori [15], as well as Benguria and Kac [16] and Ford and Kac [17], leading up to the work of Ford, Lewis, and O’Connell (FLO) [1] who, in an article entitled ”Quantum Langevin Equation”, gave a detailed discussion of the problem. In particular, these authors presented the general form of the QLE consistent with fundamental physical requirements, in particular, causality and the second law of thermodynamics. Other approaches to dissipative problems include Kubo’s linear response theory [18],[19] and the Feynman-Vernon use of path integrals which has been considerably extended by Leggett [20] and co-workers as well as many German investigators [21],[22] among others. However, the QLE approach is our method of choice since not only is it at least on a par technically with all the other approaches but we also consider it to be more physically appealing and simpler to execute.

Sec. 2 is devoted to fundamentals. In particular, we show how the QLE can be obtained from a very general and basic Hamiltonian after which we proceed to show how this leads to results for important quantum commutators and correlations which form the basis for the derivation of observable quantities. The QLE derived in Ref. 1 pertains to a stationary process, in the sense that correlations, probability distributions, etc. for the dynamical variable x are invariant under time-translation ($t \rightarrow t + t_0$). It is this equation which is the basis of most of the applications which we discuss below. There is also a QLE for the initial value problem [17] which has been used to obtain the most general master with explicit expression for the associated time-dependent coefficients, leading in turn to an explicit exact solution [23]. Sec. 3 discusses the Ohmic heat bath model and various applications and similarly for the radiation heat bath model in Sec. 4. Driven Systems are discussed in Sec. 5.

II. GENERALIZED QUANTUM LANGEVIN EQUATION

The QLE is a Heisenberg equation of motion for the coordinate operator x of a quantum particle of mass m moving in a one-dimensional potential $V(x)$ and linearly coupled to a passive heat bath at temperature T . Whereas it has a very general form, as we pointed out in Ref. 1, it can be realized with a simple and convenient model, viz., the independent-oscillator

(IO) model. The Hamiltonian of the IO system is

$$H = \frac{p^2}{2m} + V(x) + \sum_j \left(\frac{p_j^2}{2m_j} + \frac{1}{2}m_j\omega_j^2(q_j - x)^2 \right) - xf(t). \quad (2.1)$$

Here m is the mass of the quantum particle while m_j and ω_j refer to the mass and frequency of heat-bath oscillator j . In addition, x and p are the coordinate and momentum operators for the quantum particle and q_j and p_j are the corresponding quantities for the heat-bath oscillators. Also $f(t)$ is a c -number external force. The infinity of choices for the m_j and ω_j give this model its great generality. In particular, it can describe nonrelativistic quantum electrodynamics (QED), the Schwabl-Thirring model, the Ford-Kac-Mazur model, and the Lamb model [1].

Use of the Heisenberg equations of motion leads to the QLE

$$m\ddot{x} + \int_{-\infty}^t dt' \mu(t-t')\dot{x}(t') + V'(x) = F(t) + f(t), \quad (2.2)$$

where $V'(x) = dV(x)/dx$ is the negative of the time-independent external force and $\mu(t)$ is the so-called memory function. $F(t)$ is the random (fluctuation or noise) operator force with mean $\langle F(t) \rangle = 0$ and $f(t)$ is a c -number external force (due to an electric field, for instance). In addition, it should be strongly emphasized that "– the description is more general than the language –" [1] in that $x(t)$ can be a generalized displacement operator (such as the phase difference of the superconducting wave function across a Josephson junction).

Thus, the coupling with the heat bath is described by two terms: an operator-valued random force $F(t)$ with mean zero, and a mean force characterized by a memory function $\mu(t)$. Explicitly,

$$\mu(t) = \sum_j m_j \omega_j^2 \cos(\omega_j t) \theta(t), \quad (2.3)$$

, with $\theta(t)$ the Heaviside step function. Also

$$F(t) = \sum_j m_j \omega_j^2 q_j^h(t), \quad (2.4)$$

is a fluctuating operator force with mean $\langle F(t) \rangle = 0$, where $q_j^h(t)$ denotes the general solution of the homogeneous equation for the heat-bath oscillators (corresponding to no interaction). These results were used to obtain the results for the (symmetric) autocorrelation and commutator of $F(t)$, viz.,

$$\begin{aligned} C_{FF}(t-t') &= \frac{1}{2} \langle F(t)F(t') + F(t')F(t) \rangle \\ &= \frac{1}{\pi} \int_0^\infty d\omega \operatorname{Re}[\tilde{\mu}(\omega + i0^+)] \hbar \omega \coth(\hbar \omega / 2kT) \cos[\omega(t-t')], \end{aligned} \quad (2.5)$$

$$[F(t), F(t')] = \frac{2\hbar}{i\pi} \int_0^\infty d\omega \operatorname{Re}\{\tilde{\mu}(\omega + i0^+)\} \omega \sin \omega(t - t'). \quad (2.6)$$

We note that both of these results are independent of $V(x)$ and $f(t)$. Here $\tilde{\mu}(z)$ is the Fourier transform of the memory function:

$$\tilde{\mu}(z) = \int_0^\infty dt \mu(t) e^{izt}. \quad (2.7)$$

Kubo [18],[19] refers to (2.5) as the second fluctuation-dissipation theorem and we note that it can be written down explicitly once the QLE is obtained. Also, its evaluation requires only knowledge of $\operatorname{Re}\tilde{\mu}(\omega)$. On the other hand, the first fluctuation-dissipation theorem is an equation involving the autocorrelation of $x(t)$ and its explicit evaluation requires a knowledge of the generalized susceptibility $\alpha(\omega)$ (to be defined below) which is equivalent to knowing the solution to the QLE and also requires knowledge of both $\operatorname{Re}\tilde{\mu}(\omega)$ and $\operatorname{Im}\tilde{\mu}(\omega)$. This solution is readily obtained when $V(x) = 0$, corresponding to the original Brownian motion problem [2]. As shown in Ref. [24], a solution is also possible in the case of an oscillator. Taking $V(x) = \frac{1}{2}Kx^2 = \frac{1}{2}m\omega_0^2x^2$, these authors obtained the solution of the Langevin equation (2.2) in the form

$$x(t) = \int_{-\infty}^t dt' G(t - t') \{F(t') + f(t')\}, \quad (2.8)$$

where $G(t)$, the Green function, is given by

$$G(t) = \frac{1}{2\pi} \int_{-\infty}^\infty d\omega \alpha(\omega + i0^+) e^{-i\omega t}, \quad (2.9)$$

with $\alpha(z)$ the familiar response function

$$\alpha(z) = \frac{1}{-mz^2 - iz\tilde{\mu}(z) + K}. \quad (2.10)$$

Also, taking the Fourier transform of (2.8), we obtain

$$\tilde{x}(\omega) = \alpha(\omega) \{\tilde{F}(\omega) + \tilde{f}(\omega)\}, \quad (2.11)$$

where the superposed tilde is used to denote the Fourier transform. Thus, $\tilde{x}(\omega)$ is the Fourier transform of the operator $x(t)$:

$$\tilde{x}(\omega) = \int_{-\infty}^\infty dt x(t) e^{i\omega t}. \quad (2.12)$$

It is also useful to note that the commutator, which is temperature independent, is given by the formula [25]

$$[x(t_1), x(t_1 + t)] = \frac{2i\hbar}{\pi} \int_0^\infty d\omega \text{Im}\{\alpha(\omega + i0^+)\} \sin \omega t. \quad (2.13)$$

Whereas the autocorrelation and commutator of $F(t)$ are independent of $V(x)$ and $f(t)$, this is not so in the case of $x(t)$, as is obvious from (2.8) and (2.10). However, since most of our applications pertain to the case $f(t) = 0$, we will confine ourselves for the moment to this case and discuss later generalizations which are necessary when $f(t) \neq 0$. Thus, in particular, when $f(t) = 0$

$$\begin{aligned} C(t - t') &\equiv \frac{1}{2} \langle x(t)x(t') + x(t')x(t) \rangle \\ &= \frac{\hbar}{\pi} \int_0^\infty d\omega \text{Im}\{\alpha(\omega + i0^+)\} \coth \frac{\hbar\omega}{2kT} \cos \omega(t - t'). \end{aligned} \quad (2.14)$$

We note that it is a function only of the time-difference $t - t'$.

This is referred to by Kubo [18],[19] as the fluctuation-dissipation theorem of the first kind. Next, from (2.10), we see that

$$\text{Im } \alpha(\omega) = \omega |\alpha(\omega)|^2 \text{Re } \tilde{\mu}(\omega). \quad (2.15)$$

Thus, (2.14) may be expressed in the form

$$C(t) = \frac{\hbar}{\pi} \int_0^\infty d\omega \omega |\alpha(\omega)|^2 \text{Re } \tilde{\mu}(\omega) \coth \left(\frac{\hbar\omega}{2kT} \right) \cos \omega t. \quad (2.16)$$

The mean square displacement of the quantum particle in a dissipative environment, $s(t)$ say, plays a key role in many of our subsequent discussions. In particular, it determines the diffusion time through a system of interest. Thus using (2.14), we obtain

$$\begin{aligned} s(t) &\equiv \langle [x_s(t) - x_s(0)]^2 \rangle \\ &= 2 \{C(0) - C(t)\} \\ &= \frac{2\hbar}{\pi} \int_0^\infty d\omega \text{Im}\{\alpha(\omega + i0^+)\} \coth \frac{\hbar\omega}{2kT} (1 - \cos \omega t). \end{aligned} \quad (2.17)$$

For specific applications, the key ingredient to be identified is the nature of the heat bath which is simply characterized by $\text{Re } \tilde{\mu}(\omega + i0^+)$, the spectral distribution of the memory function, which is a positive real function [1] and given explicitly by the relation

$$\text{Re } [\tilde{\mu}(\omega + i0^+)] = \frac{\pi}{2} \sum_j m_j \omega_j^2 [\delta(\omega - \omega_j) + \delta(\omega + \omega_j)]. \quad (2.18)$$

Next, we summarize how the above tools have been used to obtain thermodynamic quantities, especially results which have been obtained only by the QLE method. The key quantity calculated [24] is the free energy F , which is a thermodynamic potential from which other thermodynamic functions can be obtained by differentiation.

The system of an oscillator coupled to a heat bath in thermal equilibrium at temperature T has a well-defined free energy. The free energy ascribed to the oscillator, $F(T)$, is given by the free energy of the system minus the free energy of the heat bath in the absence of the oscillator. This calculation was carried out by two different methods [24],[26] leading to the "remarkable formula":

$$F(T) = \frac{1}{\pi} \int_0^\infty d\omega f(\omega, T) \times \text{Im} \left\{ \frac{d \log \alpha(\omega + i0^+)}{d\omega} \right\}, \quad (2.19)$$

where $f(\omega, T)$ is the free energy of a single oscillator of frequency ω , given by

$$f(\omega, T) = \kappa T \log \left[1 - \exp \left(\frac{-\hbar\omega}{\kappa T} \right) \right]. \quad (2.20)$$

Here the zero-point contribution ($\frac{\hbar\omega}{2}$) has been omitted. In particular, this result was used to obtain the energy and the entropy ascribed to the oscillator.

Finally, we note that magnetic field effects on all of the above quantities have been calculated [27],[28]. In particular, it was shown that the effect of an arbitrary heat bath on Landau diamagnetism is always such as to reduce the magnitude of the magnetic moment without changing its diamagnetic character [28].

Thus, we now have all the tools necessary for applications. For each particular application, one must first decide on the correct specification of the relevant heat bath. The case of constant friction, the so-called Ohmic model, is of special interest since it is the simplest model but yet gives a good description of many physical systems. A generalization of this model, involving an additional parameter, is the single relaxation time model [23]. Also of interest is the Debye model [29]. The blackbody radiation (BBR) heat bath model [1],[24] is of fundamental interest since the relevant Hamiltonian H in this case is the universally accepted H of quantum electrodynamics. Thus, we refer to the BBR model as "the rosetta-stone of heat bath models" since it leads to readily-checked predictions and it is the basis of many of the applications which we consider below. Hence, in Sec. 3, we consider the Ohmic model and various applications and we do likewise for the BBR model in Sec. 4.

III. OHMIC HEAT BATH MODEL AND APPLICATIONS

The Ohmic model is defined by the choice

$$\mu(t) = \zeta \delta(t), \quad t > 0, \quad (3.1)$$

so that

$$\tilde{\mu}(\omega) = \zeta = m\gamma = \text{constant}. \quad (3.2)$$

Also, it follows that

$$m(\ddot{x} + \gamma\dot{x} + \omega_0^2) = F(t). \quad (3.3)$$

This corresponds to the original form of the Langevin equation except that now the classical quantities $x(t)$ and $F(t)$ are operators and we have also included an oscillator potential. Since the past motion does not appear, one says there is no memory. On the other hand, the quantum-mechanical process is not Markovian since $C_{FF}(t - t')$ is not proportional to $C_{FF}(t - t')$ a $\delta(t - t')$ except in the classical limit where $\hbar \rightarrow 0$ [1]. Using (3.3) in (2.10) leads to the result

$$\alpha(\omega) = \{m[-\omega^2 + \omega_0^2 - i\omega\gamma]\}^{-1}, \quad (3.4)$$

from which it follows, using (2.9), that the green function is given by

$$G(t) = e^{-(\frac{\gamma t}{2})} \frac{\sin \omega_1 t}{m\omega_1}, \quad (3.5)$$

where

$$\omega_1 = \{\omega_0^2 - (\frac{\gamma}{2})^2\}^{\frac{1}{2}}. \quad (3.6)$$

Also, from (2.16), we obtain

$$C(t) = \frac{\hbar\gamma}{2\pi m} \int_{-\infty}^{\infty} \frac{\omega \coth(\frac{\hbar\omega}{2kT})}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2} d\omega. \quad (3.7)$$

We now consider various applications.

A. Brownian Motion

This is the original problem of diffusion of a free particle through a medium characterized by a dissipative decay rate γ . Since here $\omega_0 = 0$, the results given above reduce to

$$\alpha(\omega) = \{m[-\omega^2 - i\omega\gamma]\}^{-1}, \quad (3.8)$$

$$G(t) = \frac{1 - e^{-\gamma t}}{m\gamma}, \quad (3.9)$$

$$C(t) = \frac{\hbar\gamma}{2\pi m} \int_{-\infty}^{\infty} d\omega \frac{\coth(\frac{\hbar\omega}{2kT})}{\omega(\omega^2 + \gamma^2)} e^{-i\omega t}. \quad (3.10)$$

Thus, (2.17) gives

$$s(t) = \frac{\hbar\gamma}{\pi m} \int_{-\infty}^{\infty} d\omega \frac{(1 - \cos \omega t)}{\omega(\omega^2 + \gamma^2)} \coth\left(\frac{\hbar\omega}{2kT}\right). \quad (3.11)$$

In the classical high temperature limit (corresponding to the Brown experiment), $\coth\left(\frac{\hbar\omega}{2kT}\right) \longrightarrow \left(\frac{2kT}{\hbar\omega}\right)$, so that

$$s(t) = \frac{2kT}{m\gamma^2} \{e^{-\gamma t} - 1 + \gamma t\}, \quad kT \gg \hbar\gamma. \quad (3.12)$$

For long times

$$s(t) \rightarrow \frac{2kT}{m\gamma} t, \quad \gamma t \gg 1, \quad (3.13)$$

which is the familiar Einstein relation from Brownian motion theory. However, for short times (which are more characteristic of decoherence decay times)

$$s(t) \rightarrow \frac{kT}{m} t^2, \quad \gamma t \ll 1, \quad (3.14)$$

independent of γ . Thus, in this simple case, the familiar Einstein diffusion coefficient

$$D \equiv \frac{1}{2} \dot{s}(t) = \frac{kT}{m\gamma}, \quad (3.15)$$

is obtained.

In particular, the situation is very different for low temperatures ($kT \ll \hbar\gamma$), in which case the main contribution is from the zero-point ($T = 0$) oscillations of the electromagnetic field. This calculation has recently been carried out [30] leading to the result

$$s(t) \cong -\frac{\hbar\zeta}{\pi m^2} t^2 \left\{ \log \frac{\zeta t}{m} + \gamma_E - \frac{3}{2} \right\}, \quad (3.16)$$

where $\gamma_E = 0.577215665$ is Euler's constant.

B. Noise in gravitational wave detector suspension systems

Crucial to the detection of gravitational waves is the mechanical system which is used to measure the relative displacements of suspended mirrors [interferometric experiments, such as the Laser Interferometric Gravitational Wave Observatory (LIGO) or the displacement of a resonant mass antenna (bar experiments)]. Here, we present a general framework for describing noise effects in detectors systems. Our results apply to a very general dissipative environment but for definiteness we will concentrate mostly on the LIGO-like test mass suspensions and on bar detectors.

The crucial quantity to be calculated is the ensemble average of the square of the displacement due to noise, $\langle x^2(t) \rangle$, which is simply equal to $C(0)$, as is clear from (2.14). Hence, from (2.16),

$$\langle x^2(t) \rangle = \frac{\hbar}{\pi} \int_0^\infty d\omega \omega |\alpha(\omega)|^2 \operatorname{Re} \tilde{\mu}(\omega) \coth \left(\frac{\hbar\omega}{2kT} \right) \equiv \int_0^\infty P(\omega) d\omega, \quad (3.17)$$

where

$$P(\omega) = \frac{\hbar}{\pi} \omega |\alpha(\omega)|^2 \operatorname{Re} \tilde{\mu}(\omega) \coth \left(\frac{\hbar\omega}{2kT} \right), \quad (3.18)$$

is the power spectrum of the coordinate fluctuations.

In the case of resonant bar detectors, $\tilde{\mu}(\omega)$ is taken to be $m\gamma$, where γ is a constant. Thus, in particular, if we take the weak-coupling limit $\omega_0 \ll \gamma$ and substitute (3.4) into (3.17), we immediately find the well-known high-temperature result $\langle x^2(t) \rangle = \frac{kT}{m\omega_0^2}$ and the zero-temperature result $\langle x^2(t) \rangle = (\frac{\hbar}{2m\omega_0})$ [31]. However, in the case of nonresonant LIGO detectors, which are responsive to a range of frequencies, the frequency dependence of $\tilde{\mu}(\omega)$ is essential. In practice, because of the complexity of the detector system, the practical procedure will be to fit the experimental results by some $\tilde{\mu}(\omega)$. This has the advantage that we know a lot about the properties of $\tilde{\mu}(\omega)$, regardless of the nature of the heat bath. In particular, it is independent of the external potential and the temperature [1]. Further details may be found in [31].

C. Resistively shunted Josephson junctions

Our purpose here is to show how some of the principal well-known theoretical results follow simply from the quantum Langevin approach and also how they may be generalized.

This will also provide an example of our statement that "– the operator x is the quantum Langevin equation – can be a generalized displacement operator –". For an ideal junction the current is given by the Josephson equation [32]:

$$I = I_C \sin \phi, \quad (3.19)$$

where ϕ is the phase difference of the superconducting wave function across the junction and I_C is the critical current. The voltage across the junction is

$$V = \frac{\hbar}{2e} \dot{\phi}, \quad (3.20)$$

where $\frac{\hbar}{2e}$ is the quantum of flux. A real junction can be viewed as a capacitance C and a shunt resistance R in parallel with an ideal junction. The current is then the sum of the ideal junction current, given by (3.19), the current through the capacitor, $\dot{Q} = C\dot{V}$, and the current through the resistor, $I = \frac{V}{R}$. The junction voltage is still given by (3.20). The basic equation of motion of the junction can therefore be written

$$\left(\frac{\hbar}{2e}\right)^2 C \ddot{\phi} + \left(\frac{\hbar}{2e}\right)^2 \frac{\dot{\phi}}{R} + \frac{\hbar}{2e} I_C \sin \phi = \frac{\hbar}{2e} I + F(t). \quad (3.21)$$

This is of the form of a quantum Langevin equation with mass and friction constant

$$m = \left(\frac{\hbar}{2e}\right)^2 C, \quad \zeta = \frac{\left(\frac{\hbar}{2e}\right)^2}{R}, \quad (3.22)$$

and with potential

$$U(\phi) = -\frac{\hbar}{2e} (I\phi + I_C \cos \phi). \quad (3.23)$$

Thus, analogous to (3.17), the ensemble average of the square of the phase due to noise is

$$\langle \phi^2 \rangle = \frac{4e^2}{\pi C \hbar} \int_0^\infty d\omega \coth \left(\frac{\hbar \omega}{2kT} \right) \frac{\omega \gamma}{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2}, \quad (3.24)$$

where

$$\gamma = \frac{1}{RC}, \quad \omega_0^2 = \frac{2e}{C \hbar} (I_C^2 - I^2)^{\frac{1}{2}}. \quad (3.25)$$

In the limit of large shunt resistance (weak coupling limit), $\gamma \ll \omega_0$, this becomes

$$\langle \phi^2 \rangle = \frac{2e^2}{C \hbar \omega_0} \coth \left(\frac{\hbar \omega_0}{2kT} \right). \quad (3.26)$$

This weak coupling limit corresponds to the expression for the phase fluctuations obtained by Josephson [33]. The power spectrum of the voltage fluctuations is readily obtained from (3.24) using (3.20). Further details may be found in [34].

D. Environmental Effects on Nanosystems

The study of nanosystems is now burgeoning and, because of the small dimensions involved, environmental effects play an important role. To illustrate, we will focus our discussion on the analysis of charge fluctuations on small-capacitance tunnel junctions. Following Ingold and Nazarov [35], we consider a junction of capacitance C carrying the charge $Q = CV$ where V is the voltage across the junction. The external circuit (the environment) is described by its impedance $Z(\omega) = \frac{V(\omega)}{I(\omega)}$, where $I(\omega)$ is the current. The phase difference across the junction is obtained from (3.20) [except the $2e \rightarrow e$ since here we are not considering a superconducting tunnel junction] to get

$$\phi(t) = \frac{e}{\hbar} \int_{-\infty}^t dt' V(t'). \quad (3.27)$$

Turning to a quantum picture, it follows that $(\frac{\hbar}{e})\phi$ and Q are the electrical equivalent of the mechanical quantities x and p (see table I of Ref. 35 for more details). It readily follows that there is a fluctuation-dissipation theorem describing the relation between the charge fluctuation $q(t)$ and the dissipation described by $Z(\omega)$. Explicitly, we have [36]

$$\langle q^2(t) \rangle = \int_0^\infty d\omega \frac{\hbar\omega C^2}{\pi} \coth \frac{\hbar\omega}{2k_B T} \text{Re} \left[\frac{1}{i\omega C + Z^{-1}(\omega)} \right]. \quad (3.28)$$

This result is applicable to any kind of external circuit attached to a small junction. In particular, this result was used to calculate the effect of quantum smearing of Coulomb blockade in small tunnel junctions [37] and it was shown that, in the weak coupling limit, the QLE theory is related to the well-known generalized Landauer formula [38]. Finally, we note that big efforts are currently being made toward developing single charge transfer devices and, in that context, we have used (3.27) to calculate environmental effects on a single electron box [39].

E. Quantum Transport for a many-body System

The problem of electrical conductivity in solids has been studied by a large number of theoretical methods. For the most part, they are either based on the Boltzmann transport equation for the distribution of electrons, or they start with the Kubo formula in linear-response theory. In spite of their successes, those two methods are known to be impractical

tools for calculating the electrical transport properties of systems with high-order impurity scattering, among other disadvantages. Thus, we were motivated to treat the quantum transport of an interacting system of electrons, impurities, and phonons, in a time-dependent electric field, by using the QLE in which the system is shown to be equivalent to a quantum particle in a heat bath [40]. The center of mass of the electrons acts like a quantum particle, while the relative electrons and phonons play the role of a heat bath. They are coupled through the electron-impurity and electron-phonon interactions. After eliminating the heat-bath variables, the equation of motion for the quantum particle is written in a form of a QLE, with a memory term which reflects the retarded effects of the heat bath on the quantum particle. The evaluation of the memory term immediately leads to a result for the susceptibility from which we can calculate the conductivity directly, in contrast to Kubo-type calculations which require the evaluation of correlation functions as an intermediate step. Our results were then applied successfully to transport in 3, 2, 1, and 0 dimensional systems and we refer to our review [41] for details. As emphasized in [41], a big advantage of the QLE approach is that it gives a natural separation between the conductivity and the noise.

F. Unruh radiation

It is generally accepted that a system that undergoes uniform acceleration with respect to the vacuum of flat-space-time will thermalize at a temperature (the so-called Unruh temperature) that is proportional to the acceleration [42]. However, the question of whether or not the system actually radiates is highly controversial. Thus, we were motivated to carry out an exact analysis of the problem using a generalized quantum Langevin equation to describe the motion of an oscillator (the detector) moving under a constant force and coupled to a one-dimensional scalar field (scalar electrodynamics). We conclude that the system does not radiate despite the fact that it does in fact thermalize at the Unruh temperature [43]. The essence of the calculation was to show that the equation of motion of the oscillator has the same form as that given in (3.3) except that now the autocorrelation given in (2.5) not only corresponds to an Ohmic heat bath but the relevant temperature is the Unruh temperature which, of course, is proportional to the acceleration. However, due to the fact that there is no term analogous to the external force $f(t)$ appearing (2.2), it immediately

follows that there is no radiation, as confirmed by a detailed calculation. What we found is that the uniform acceleration does not give rise to a constant force in the equation of motion but, instead, it only appears explicitly in the autocorrelation of the fluctuation force. In other words, in this case, the constant external acceleration does not cause radiation but simply accentuates the noise.

IV. BLACKBODY RADIATION HEAT BATH MODEL AND APPLICATIONS

The relevant Hamilton H is the universally accepted H of quantum electrodynamics (QED) from which it follows [1],[24] that

$$Re[\tilde{\mu}(\omega + i0^+)] = \frac{2e^2\omega^2}{3c^3}f_k^2, \quad (4.1)$$

where the quantity f_k is the electron form factor (Fourier transform of the electron charge distribution). In other words, we have allowing the electron to have structure.

The physically significant results for this model should not depend upon details of the electron form factor, subject, of course, to the condition that is be unity up to some large frequency Ω and falls to zero thereafter. A convenient form which satisfies this condition is

$$f_k^2 = \frac{\Omega^2}{\omega^2 + \Omega^2}. \quad (4.2)$$

Using this in (4.1), the Stieltjes inversion formula gives

$$\tilde{\mu}(z) = \frac{2e^2\Omega^2}{3c^3} \frac{z}{z + i\Omega}. \quad (4.3)$$

We see here a manifestation of the general feature that the memory function is independent of the external potential and the particle mass.

As essential aspect of QED theory is the necessity for mass renormalization. Thus the m occurring in the QLE is actually the bare mass and the renormalized (observed) mass M is given in terms of the bare mass m by the relation

$$M = \frac{m + 2e^2\Omega}{3c^3} = m + \tau_e\Omega M, \quad (4.4)$$

where

$$\tau_e = \frac{2e^2}{3Mc^3} \simeq 6 \times 10^{-24} \text{ sec}. \quad (4.5)$$

Next, taking the inverse Fourier transform of (4.3) leads to the result [44]

$$\mu(t) = M\Omega^2\tau_e[2\delta(t) - \Omega \exp(-\Omega t)], \quad (4.6)$$

which makes manifest the non-Markovian nature of the motion. We now consider various applications.

A. Elimination of Runaway Solutions in Electron Radiation Theory

The fact that $\tilde{\mu}(\omega)$ is a positive function (which follows from the second law of thermodynamics) is equivalent to the demand that all the poles of $\alpha(\omega)$, the generalized susceptibility, must lie in the lower half of the complex plane (which also follows from the principle of causality). This implies that $m \geq 0$ and that

$$\Omega \leq \tau_e^{-1} = 1.6 \times 10^{23} \text{ s}^{-1}. \quad (4.7)$$

If one selects for Ω its largest permissible value: namely τ_e^{-1} (corresponding to letting $m \rightarrow 0$, which is also equivalent to choosing the closest approach to a point electron consistent with causality), then one obtains the classical equation of motion

$$M\ddot{x} = f(t) + \tau_e \dot{f}(t). \quad (4.8)$$

This result has a variety of desirable properties not exhibited by the Abraham-Lorentz equation. Generalizations which include quantum effects and the presence of a potential may be found in [44],[45]. The Larmor formula also requires generalization [46],[47]. A relativistic generalization of (4.8) is given in [48] along with a proof that a constant force does not cause radiation.

B. Rydberg Atom Level Shifts due to Blackbody Radiation

Atoms coupled to a bath in thermal equilibrium at temperature T are best described in terms of the free energy [24] and we have argued [49],[50] that this is the quantity actually measured by experiment [51]. Thus, using (2.19), we found that, in the high temperature limit, where kT is large compared with the level spacing, the shift in free energy is given by

$$\Delta F = \frac{\pi e^2 (kT)^2}{9\hbar M c^3}, \quad (4.9)$$

in agreement with the experimental results [51].

V. DECOHERENCE

The superposition principle (and related work on Schrödinger cats), entanglement, and the quantum-classical interface are at the cutting-edge of topical research [52]. Since superposition states are very sensitive to decoherence, reservoir theory has attracted much recent interest. Recognizing that conventional master equation approaches are often not adequate, some investigators have used path integral methods. However, we have found that the simplest and most physically appealing approach to the problem is via use of the QLE, supplemented by use of the Wigner distribution function (WDF) for the study of a system with an infinite spectrum of states [53] and by use of the spin polarization vector in the case of spin systems [54]. For the investigation of decoherence phenomena, we have found that it is important to distinguish between two different physical scenarios: namely (a) complete entanglement between the quantum particle and the heat bath at all times [55] and (b) the system in a state in which the oscillator is not coupled to the bath at, say, $t = 0$ and such that the bath is in equilibrium at temperature T [56]. Thus, it takes a characteristic time of the order of γ^{-1} (where γ is a typical dissipative decay time) for the complete coupling to occur and for the whole system to come into thermal equilibrium. It is noteworthy that "decoherence without dissipation" can occur [55-57] but not for low temperatures [58].

Scenario (a), the "entanglement at all times" calculation utilized quantum probability distributions (which are related to Wigner distributions) in conjunction with results obtained by use of the stationary solution to the QLE, given in (2.2). On the other hand, for the scenario (b) case, it is necessary to take into account initial conditions. The Langevin equation for the oscillator with given initial values is given by [17],[56]

$$m\ddot{x} + \int_0^t dt' \mu(t-t')\dot{x}(t') + Kx = -\mu(t)x(0) + F(t), \quad (5.1)$$

and the general solution is given by

$$x(t) = m\dot{G}(t)x(0) + mG(t)\dot{x}(0) + X(t), \quad (5.2)$$

where we have introduced the fluctuating position operator,

$$X(t) = \int_0^t dt' G(t-t')F(t'). \quad (5.3)$$

If we assume that at $t = 0$ the system is in a state in which the oscillator is not coupled to the bath and that the bath is in equilibrium at temperature T , we find that the correlation

and commutator are the same as those for the stationary equation. It was then possible to show [56] that one could write the Langevin equation (5.1) in the form of an equation that is local in time with time-dependent coefficients:

$$\ddot{x} + 2\Gamma(t)\dot{x} + \Omega^2(t)x = \frac{1}{m}F(t), \quad (5.4)$$

where explicit expressions for $\Gamma(t)$ and $\Omega(t)$ were obtained in terms of $G(t)$. Furthermore, it was shown that these results constitute in essence a derivation of the HPZ exact master equation [59] with explicit expressions for the time-dependent coefficients. It is also notable that in this scenario one needs to assume that the initial temperature of the particle is the same as that of the heat bath in order to obtain "decoherence without dissipation".

VI. DRIVEN SYSTEMS

As remarked in Sec. II, the position autocorrelation, as distinct from the fluctuation force autocorrelation, is modified by a presence of an external force $f(t)$, as discussed in detail in [60],[61]. It is convenient to now write

$$x(t) = x_s(t) + x_d, \quad (6.1)$$

and

$$C(t) = C_0(t) + C_d(t), \quad (6.2)$$

where x_d is the "driven" contribution due to the external force $f(t)$ and x_s is the contribution due to the fluctuation force $F(t)$. Here we have introduced a subscript s to emphasize that $x_s(t)$ is a stationary operator-process, in the sense that correlations, probability distributions, etc. for this dynamical variable are invariant under time-translation ($t \rightarrow t + t_0$). In particular, the correlation $C_0(t)$ is given by the right-side of (2.14) and is a function only of the time-difference $t - t'$. Furthermore, [61]

$$C_d(t, t') = \langle x(t) \rangle \langle x(t') \rangle, \quad (6.3)$$

where $\langle x(t) \rangle$ is the steady mean of the driven motion. Based on these results, we extended the calculation of the well-known Burshtein-Mollow spectrum of resonance fluorescence to the case of non-zero temperature. In the high temperature limit, $\kappa T \gg \hbar\omega_0$, where $\hbar\omega_0$ is the resonant energy for the two-level (atom), we found that the decay rate is increased

by a factor $\left(\frac{\kappa T}{\hbar\omega_0}\right)$. Our calculation was based on use of the Lax formula for two-time correlations but, as we also pointed out, this formula is applicable only for weak coupling and for frequencies near a resonance frequency since, more generally, the Onsager classical regression theorem cannot be generalized to the quantum domain [61],[62].

A further application of these results was the consideration of decoherence phenomena in the presence of an external field [63],[64]. This work was motivated by the recent interest in engineered reservoirs [65]. In particular, Myatt et al. [66] used a linear Paul trap to confine single Be ions in a harmonic potential and then prepared various superposition states. Next, they induced decoherence by coupling the single ion to a reservoir which they controlled in various ways. Such a reservoir gives rise to an external force $f(t)$ in the equation of motion of the system. The calculations in [63],[64] led to the conclusion that a non-random external field does not give rise to decoherence whereas, by contrast, a random field does. In particular, the experiments of Myatt et al. [66] used a δ -correlated force. The existing experiments verified the familiar result that the decoherence decay time τ_d is inversely proportional to the square of the separation d of the superposition components. However, this is a familiar result predicted by the plethora of papers dealing with the $f(t) = 0$ situation but it does not give information on the dependence of τ_d on the parameters of the externally-superimposed reservoir. Thus, we will have to await come experimental data in order to compare with existing theory.

VII. ACKNOWLEDGEMENT

The author is pleased to acknowledge that all of the essential results described above were derived in collaboration with Prof. G. W. Ford.

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